

Limites contenant des fonctions trigonométriques

Rappels : $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$ $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$

* $\lim_{x \rightarrow 0} \frac{\sin 2x}{3x} = ?$ $\left. \begin{array}{l} \lim_{x \rightarrow 0} \sin 2x = 0 \\ \lim_{x \rightarrow 0} 3x = 0 \end{array} \right\}$ On a donc une forme indéterminée %

Pour lever l'indétermination, on fait apparaître une limite de cosinus connue

$\lim_{x \rightarrow 0} \frac{\sin 2x}{3x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times \frac{2x}{3x}$ On pose $X = 2x$ si $x \rightarrow 0$ $X \rightarrow 0$

Donc $\lim_{x \rightarrow 0} \frac{\sin 2x}{3x} = \lim_{X \rightarrow 0} \frac{\sin X}{X} \times \frac{2}{3} = 1 \times \frac{2}{3} = \frac{2}{3}$

* $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = ?$ $\left. \begin{array}{l} \lim_{x \rightarrow 0} \sin 5x = 0 \\ \lim_{x \rightarrow 0} 3x = 0 \end{array} \right\}$ On a une forme indéterminée %

Donc changement de variable $X = 5x$ si $x \rightarrow 0$, $X \rightarrow 0$

Donc $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times \frac{5x}{3x} = \lim_{X \rightarrow 0} \frac{\sin X}{X} \times \frac{5}{3} = 1 \times \frac{5}{3} = \frac{5}{3}$

* $\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan x} = ?$ $\left. \begin{array}{l} \lim_{x \rightarrow 0} \sin 3x = 0 \\ \lim_{x \rightarrow 0} \tan x = 0 \end{array} \right\}$ Forme indéterminée %

On a $\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times \frac{3x}{x} \times \frac{x}{\tan x}$
 $= \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times 3 \times \frac{1}{\frac{\tan x}{x}}$ En posant $X = 3x$ si $x \rightarrow 0$ $X \rightarrow 0$

On obtient $\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan x} = \lim_{X \rightarrow 0} \frac{\sin X}{X} \times 3 \times \frac{1}{\lim_{x \rightarrow 0} \frac{\tan x}{x}} = 1 \times 3 \times \frac{1}{1} = 3$

On a fait apparaître des limites connues du cours pour pouvoir déterminer les limites cherchées

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sqrt{1-\cos 2x}} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times \frac{3x}{\sqrt{1-\cos 2x}} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \times \sqrt{\frac{(3x)^2}{1-\cos 2x}}$$

En posant $x=3x$ si $x \rightarrow 0$ $x \rightarrow 0$ On obtient la limite suivante

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sqrt{1-\cos 2x}} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \times \frac{1}{\sqrt{\frac{1-\cos x}{x^2}}} = 1 \times \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2 \sin x + 3 \tan x}{\sqrt{8x^2 - 2x^3}} &= ? & \lim_{x \rightarrow 0} 2 \sin x + 3 \tan x &= 0 \\ \lim_{x \rightarrow 0} \sqrt{8x^2 - 2x^3} &= 0 \end{aligned} \quad \left. \vphantom{\lim_{x \rightarrow 0} \frac{2 \sin x + 3 \tan x}{\sqrt{8x^2 - 2x^3}}} \right\} \text{Forme indéterminée } 0/0$$

On transforme l'écriture $\lim_{x \rightarrow 0} \frac{2 \sin x + 3 \tan x}{\sqrt{8x^2 - 2x^3}} = \lim_{x \rightarrow 0} \frac{2 \sin x + 3 \tan x}{x} \times \frac{x}{\sqrt{2} \sqrt{8-2x}}$

$$= \lim_{x \rightarrow 0} \left(\frac{2 \sin x}{x} + \frac{3 \tan x}{x} \right) \times \frac{x}{\sqrt{2} \sqrt{8-2x}}$$

si $x < 0$ $\lim_{x \rightarrow 0} \left(\frac{2 \sin x}{x} + \frac{3 \tan x}{x} \right) \times \frac{(-1)}{\sqrt{8-2x}} = (2+3) \times \frac{(-1)}{2\sqrt{2}} = \frac{-5}{2\sqrt{2}} = \frac{-5\sqrt{2}}{4}$

si $x > 0$ $\lim_{x \rightarrow 0} \left(\frac{2 \sin x}{x} + \frac{3 \tan x}{x} \right) \times \frac{1}{\sqrt{8-2x}} = (2+3) \times \frac{1}{2\sqrt{2}} = \frac{5}{2\sqrt{2}} = \frac{5\sqrt{2}}{4}$

$$\begin{aligned} * \lim_{x \rightarrow \pi/2} \frac{\cos 5x}{\cos 7x} & \quad \lim_{x \rightarrow \pi/2} \cos 5x = 0 \\ & \quad \lim_{x \rightarrow \pi/2} \cos 7x = 0 \end{aligned} \quad \left. \vphantom{\lim_{x \rightarrow \pi/2} \frac{\cos 5x}{\cos 7x}} \right\} \text{On a une forme indéterminée } 0/0$$

On transforme l'écriture en posant $x = \pi/2 - x$ si $x \rightarrow \pi/2$ $x \rightarrow 0$ $x = \pi/2 - x$

Donc $\lim_{x \rightarrow \pi/2} \frac{\cos 5x}{\cos 7x} = \lim_{x \rightarrow 0} \frac{\cos 5(\pi/2 - x)}{\cos 7(\pi/2 - x)} = \lim_{x \rightarrow 0} \frac{\cos (5\pi/2 - 5x)}{\cos (7\pi/2 - 7x)}$

$$= \lim_{x \rightarrow 0} \frac{\cos 5\pi/2 \cos 5x + \sin 5\pi/2 \sin 5x}{\cos 7\pi/2 \cos 7x + \sin 7\pi/2 \sin 7x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{-\sin 7x} = \lim_{x \rightarrow 0} -\frac{\sin 5x}{5x} \times \frac{5x}{7x} \times \frac{7x}{\sin 7x}$$

$$= \lim_{x \rightarrow 0} -\frac{\sin 5x}{5x} \times \frac{5}{7} \times \frac{1}{\frac{\sin 7x}{7x}} \quad \text{Ce qu'on donne d'après le premier}$$

$$\lim_{x \rightarrow \pi/2} \frac{\cos 5x}{\cos 7x} = \lim_{x \rightarrow 0} -\frac{\sin 5x}{5x} \times \frac{5}{7} \times \frac{1}{\frac{\sin 7x}{7x}} = -1 \times \frac{5}{7} \times \frac{1}{1} = \frac{-5}{7}$$

Rappel $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

(formules d'addition)

$$* \lim_{x \rightarrow 0} \frac{\tan 2x}{\tan 5x} = ? \quad \left. \begin{array}{l} \lim_{x \rightarrow 0} \tan 2x = 0 \\ \lim_{x \rightarrow 0} \tan 5x = 0 \end{array} \right\} \text{On a une forme} \\ \text{indéterminée } 0/0$$

$$\lim_{x \rightarrow 0} \frac{\tan 2x}{\tan 5x} = \lim_{x \rightarrow 0} \frac{\tan 2x}{2x} \times \frac{2x}{5x} \times \frac{5x}{\tan 5x} \quad \begin{array}{l} \text{En posant } x=2x \text{ et } y=5x \\ \text{si } x \rightarrow 0 \text{ } x \rightarrow 0 \text{ et } y \rightarrow 0 \end{array}$$

$$\text{On obtient } \lim_{x \rightarrow 0} \frac{\tan 2x}{\tan 5x} = \lim_{x \rightarrow 0} \frac{\tan x}{x} \times \frac{2}{5} \times \lim_{y \rightarrow 0} \frac{1}{\tan y} = 1 \times \frac{2}{5} \times 1 = \frac{2}{5}$$

$$* \lim_{x \rightarrow 0} \frac{1 - \cos x}{3 \sin 2x} = ? \quad \left. \begin{array}{l} \lim_{x \rightarrow 0} 1 - \cos x = 0 \\ \lim_{x \rightarrow 0} 3 \sin 2x = 0 \end{array} \right\} \text{C'est une forme} \\ \text{indéterminée } 0/0$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{3 \sin 2x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \frac{1}{3} \times \frac{x^2}{\sin 2x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \frac{1}{3} \times \frac{1}{\frac{\sin 2x}{2x}}$$

$$\text{D'après le cours } \lim_{x \rightarrow 0} \frac{1 - \cos x}{3 \sin 2x} = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{1} \times \frac{0}{2} = 0$$

$$* \lim_{x \rightarrow 0} \frac{1 - \cos x}{7 \sin^2 x} = ? \quad \left. \begin{array}{l} \lim_{x \rightarrow 0} 1 - \cos x = 0 \\ \lim_{x \rightarrow 0} 7 \sin^2 x = 0 \end{array} \right\} \text{On a encore une forme} \\ \text{indéterminée } 0/0$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{7 \sin^2 x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \frac{1}{7} \times \left(\frac{x}{\sin x} \right)^2 = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \times \frac{1}{7} \times \frac{1}{\left(\frac{\sin x}{x} \right)^2}$$

$$\text{Ce qu'on donne d'après le cours } \lim_{x \rightarrow 0} \frac{1 - \cos x}{7 \sin^2 x} = \frac{1}{2} \times \frac{1}{7} \times \frac{1}{(1)^2} = \frac{1}{14}$$

$$* \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin^2 x} = ? \quad \left. \begin{array}{l} \lim_{x \rightarrow 0} \sin 2x = 0 \\ \lim_{x \rightarrow 0} \sin^2 x = 0 \end{array} \right\} \text{Forme indéterminée} \\ 0/0$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times \frac{2x}{x^2} \times \frac{x^2}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \times \frac{2}{x} \times \frac{1}{\left(\frac{\sin x}{x} \right)^2}$$

$$\text{Ce qu'on donne d'après le cours } \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin^2 x} = 1 \times \infty \times \frac{1}{(1)^2} = +\infty$$

$$\begin{array}{l}
 * \lim_{x \rightarrow \pi/2} (\cos x - \sin x) \tan x = ? \\
 \lim_{x \rightarrow \pi/2} \cos x - \sin x = 0 \\
 \lim_{x \rightarrow \pi/2} \tan x = +\infty
 \end{array}
 \left. \vphantom{\begin{array}{l} * \lim_{x \rightarrow \pi/2} (\cos x - \sin x) \tan x = ? \\ \lim_{x \rightarrow \pi/2} \cos x - \sin x = 0 \\ \lim_{x \rightarrow \pi/2} \tan x = +\infty \end{array}} \right\} \begin{array}{l} \text{Forme} \\ \text{indéterminée} \\ 0 \times \infty \end{array}$$

On fait un changement de variable le posant $x = \pi/2 - x \Leftrightarrow x = \pi/2 - x$ quand $x \rightarrow \pi/2$ $x \rightarrow 0$ On va donc transformer l'écriture pour cette limite

$$\begin{aligned}
 \lim_{x \rightarrow \pi/2} (\cos x - \sin x) \tan x &= \lim_{x \rightarrow 0} \left(\cos\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - x\right) \right) \tan\left(\frac{\pi}{2} - x\right) \\
 &= \lim_{x \rightarrow 0} \frac{\left(\cos\frac{\pi}{4} \cos x + \sin\frac{\pi}{4} \sin x - \left(\sin\frac{\pi}{4} \cos x - \cos\frac{\pi}{4} \sin x \right) \right) \sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} \\
 &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \sin x - \frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \sin x \right) (\sin x)}{\sin x} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt{2} \sin x}{\sin x} = \lim_{x \rightarrow 0} \sqrt{2} \frac{\sin x}{x} \times \frac{x}{2} \times \cos x \times \frac{1}{\frac{\sin x}{x} \times x} \\
 &= \lim_{x \rightarrow 0} \sqrt{2} \frac{\sin x}{x} \times \frac{1}{2} \times \cos x \times \frac{1}{\frac{\sin x}{x}} = \sqrt{2} \times 1 \times \frac{1}{2} \times 1 \times \frac{1}{1} = \frac{\sqrt{2}}{2}
 \end{aligned}$$

d'après les limites connues du cours

$$\begin{array}{l}
 * \lim_{x \rightarrow 0} x \left(\sin\left(\frac{1}{x}\right) - \frac{1}{\sin x} \right) = ? \\
 \lim_{x \rightarrow 0} x = 0 \\
 \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) - \frac{1}{\sin x} = -\infty
 \end{array}
 \left. \vphantom{\begin{array}{l} * \lim_{x \rightarrow 0} x \left(\sin\left(\frac{1}{x}\right) - \frac{1}{\sin x} \right) = ? \\ \lim_{x \rightarrow 0} x = 0 \\ \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) - \frac{1}{\sin x} = -\infty \end{array}} \right\} \begin{array}{l} \text{Forme} \\ \text{indéterminée} \\ 0 \times \infty \end{array}$$

Rappel $\forall y \in \mathbb{R} -1 \leq \sin y \leq 1$

$$\lim_{x \rightarrow 0} x \left(\sin\left(\frac{1}{x}\right) - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) - \frac{x}{\sin x} = \lim_{x \rightarrow 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} - \frac{1}{\left(\frac{\sin x}{x}\right)}$$

$$\lim_{x \rightarrow 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 0 \text{ car } \lim_{x \rightarrow 0} \left| \sin\left(\frac{1}{x}\right) \right| \leq 1 \text{ et que } \lim_{x \rightarrow 0} \frac{1}{x} = \infty$$

$$\text{comme } \lim_{x \rightarrow 0} \frac{-1}{\frac{\sin x}{x}} = -1$$

$$\text{On en déduit que } \lim_{x \rightarrow 0} x \left(\sin\left(\frac{1}{x}\right) - \frac{1}{\sin x} \right) = -1$$

$$* \lim_{x \rightarrow 0} \frac{x(1-\cos x)}{\sin^3 x - 3\sin x} = ? \quad \left. \begin{array}{l} \lim_{x \rightarrow 0} x(1-\cos x) = 0 \\ \lim_{x \rightarrow 0} \sin^3 x - 3\sin x = 0 \end{array} \right\} \text{c'est une forme indéterminée } \frac{0}{0}$$

On transforme l'écriture $1-\cos x = 2\sin^2 \frac{x}{2}$

$$\sin^3 x - 3\sin x = \sin(x+2x) - 3\sin x = \sin x \cos 2x + \sin 2x \cos x - 3\sin x$$

Comme $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ et $\sin 2x = 2\sin x \cos x$ on obtient

$$\sin^3 x - 3\sin x = \sin x(1-2\sin^2 x) + 2\sin x \cos^2 x - 3\sin x \quad \text{ce qui donne}$$

$$\begin{aligned} \sin^3 x - 3\sin x &= \sin x - 2\sin^3 x + 2\sin x(1-\sin^2 x) - 3\sin x \\ &= \sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x - 3\sin x \\ &= -4\sin^3 x \end{aligned}$$

$$\text{On a donc } \lim_{x \rightarrow 0} \frac{x(1-\cos x)}{\sin^3 x - 3\sin x} = \lim_{x \rightarrow 0} \frac{x \times 2\sin^2(\frac{x}{2})}{-4\sin^3 x} = \lim_{x \rightarrow 0} -\frac{x \sin^2(\frac{x}{2})}{2\sin^3 x}$$

$$= \lim_{x \rightarrow 0} -\frac{1}{2} \times x \times \frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2} \times \left(\frac{x}{2}\right)^2 \times \frac{1}{\frac{\sin^3 x}{x^3} \times x^3}$$

$$= \lim_{x \rightarrow 0} -\frac{1}{2} \times \frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2} \times \frac{1}{4} x^3 \times \frac{1}{\left(\frac{\sin x}{x}\right)^3 x^3}$$

$$= \lim_{x \rightarrow 0} -\frac{1}{2} \left(\frac{\sin(\frac{x}{2})}{(\frac{x}{2})}\right)^2 \times \frac{1}{4} \times \frac{1}{\left(\frac{\sin x}{x}\right)^3} = -\frac{1}{2} \times 1^2 \times \frac{1}{4} \times \frac{1}{1^3} = -\frac{1}{8}$$

d'après les limites du cours

$$* \lim_{x \rightarrow 0} \frac{1}{2(1-\cos x)} - \frac{1}{\sin^2 x} \quad \left. \begin{array}{l} \lim_{x \rightarrow 0} \frac{1}{2(1-\cos x)} = +\infty \\ \lim_{x \rightarrow 0} -\frac{1}{\sin^2 x} = -\infty \end{array} \right\} \text{Forme indéterminée } \infty - \infty$$

$$\lim_{x \rightarrow 0} \frac{1}{2(1-\cos x)} - \frac{1}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{1}{2 \times 2 \sin^2(\frac{x}{2})} - \frac{1}{4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}$$

$$\text{car } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \quad \text{et } 1-\cos x = 2 \sin^2 \frac{x}{2}$$

$$\text{donc } \lim_{x \rightarrow 0} \frac{1}{2(1-\cos x)} - \frac{1}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{1}{4 \sin^2(\frac{x}{2})} \left(1 - \frac{1}{\cos^2 \frac{x}{2}}\right)$$

$$= \lim_{x \rightarrow 0} \frac{1}{4} \times \frac{1}{\sin^2(\frac{x}{2})} \left(\frac{\cos^2(\frac{x}{2}) - 1}{\cos^2(\frac{x}{2})}\right) = \lim_{x \rightarrow 0} \frac{1}{4} \times \frac{1}{\sin^2(\frac{x}{2})} \left(\frac{-\sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2})}\right)$$

$$= \lim_{x \rightarrow 0} -\frac{1}{4} \times \frac{1}{\cos^2(\frac{x}{2})} = -\frac{1}{4}$$

$$* \lim_{x \rightarrow \pi/4} \frac{1-2\sin^2 x}{1+\cos 4x} = ?$$

$$\left. \begin{array}{l} \lim_{x \rightarrow \pi/4} 1-2\sin^2 x = 0 \\ \lim_{x \rightarrow \pi/4} 1+\cos 4x = 0 \end{array} \right\} \begin{array}{l} \text{Forme} \\ \text{indéterminée} \\ 0/0 \end{array}$$

On transforme l'expression en faisant un changement de variable $X = x - \pi/4$ quand $x \rightarrow \pi/4$ $X \rightarrow 0$

$$\text{On a donc } \lim_{x \rightarrow \pi/4} \frac{1-2\sin^2 x}{1+\cos 4x} = \lim_{X \rightarrow 0} \frac{1-2\sin^2 (X+\pi/4)}{1+\cos (4X+\pi)}$$

$$\text{Or } 1-\cos 2a = 2\sin^2 a \quad (\Leftrightarrow) \quad 1-2\sin^2 a = \cos 2a \quad \text{d'où}$$

$$1-2\sin^2 (X+\pi/4) = \cos (2X+\pi/2) \quad \text{De même } 1+\cos 2a = 2\cos^2 a$$

$$\text{donc } 1+\cos (4X+\pi) = 2\cos^2 (2X+\pi/2) \quad \text{Par conséquent}$$

$$\lim_{X \rightarrow 0} \frac{1-2\sin^2 (X+\pi/4)}{1+\cos (4X+\pi)} = \lim_{X \rightarrow 0} \frac{\cos (2X+\pi/2)}{2\cos^2 (2X+\pi/2)} = \lim_{X \rightarrow 0} \frac{1}{2\cos (2X+\pi/2)}$$

$$\text{et } \lim_{X \rightarrow 0} \frac{1}{2\cos (2X+\pi/2)} = +\infty$$

$$* \lim_{x \rightarrow \pi/2} \frac{1-\sin x - \cos x}{1-\sin x + \cos x} = ?$$

$$\left. \begin{array}{l} \lim_{x \rightarrow \pi/2} 1-\sin x - \cos x = 0 \\ \lim_{x \rightarrow \pi/2} 1-\sin x + \cos x = 0 \end{array} \right\} \begin{array}{l} \text{Forme} \\ \text{indéterminée } 0/0 \end{array}$$

On fait le changement de variable suivant $X = \pi/2 - x$ quand $x \rightarrow \pi/2$ $X \rightarrow 0$

$$\lim_{x \rightarrow \pi/2} \frac{1-\sin x - \cos x}{1-\sin x + \cos x} = \lim_{X \rightarrow 0} \frac{1-\sin (\pi/2 - X) - \cos (\pi/2 - X)}{1-\sin (\pi/2 - X) + \cos (\pi/2 - X)} = \lim_{X \rightarrow 0} \frac{1-\cos X - \sin X}{1-\cos X + \sin X}$$

$$= \lim_{X \rightarrow 0} \frac{2\sin^2 X/2 - 2\sin X/2 \cos X/2}{2\sin^2 X/2 + 2\sin X/2 \cos X/2}$$

$$\text{car } 1-\cos X = 2\sin^2 X/2$$

$$\text{et } \sin X = 2\sin X/2 \cos X/2$$

$$= \lim_{X \rightarrow 0} \frac{\sin X/2 - \cos X/2}{\sin X/2 + \cos X/2} = \frac{0-1}{0+1} = -1$$